INDEFINITE INTEGRATION

BASIC THEOREMS ON INTEGRATION

If f(x), g(x) are two functions of a variable x and k is a constant, then

(i) $\int k\, f(x)\, dx = k \int f(x)\, dx$

(ii) $\int [f(x) \pm g(x)]\; dx = \int f(x)\, dx \pm \int g(x)\, dx$

(iii) $\frac{d}{dx}\left(\int f(x)\, dx\right) = f(x)$

(iv) $\int \left(\frac{d}{dx} f(x)\right) dx = f(x) + c$

SOME STANDARD FORMULAE

(i) $\int x^n dx = \frac{x^{n+1}}{n+1} + c\ (n \neq -1)$

(ii) $\int \frac{1}{x} dx = \log_e |x| + c$

(iii) $\int e^x dx = e^x + c$

(iv) $\int a^x dx = \frac{a^x}{\log_e a} + c = a^x \log_a e + c$

(v) $\int \sin x\, dx = -\cos x + c$

(vi) $\int \cos x\, dx = \sin x + c$

(vii) $\int \tan x\; dx = \log|\sec x| + c = -\log|\cos x| + c$

(viii) $\int \cot x\; dx = \log|\sin x| + c$

(ix) $\int \sec x\; dx = \log|\sec x + \tan x| + c = -\log|\sec x - \tan x| + c$ $= \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + c$

(x) $\int \text{cosec}x\; dx = -\log|\text{cosec}\; x + \cot\; x| = \log|\text{cosec}x - \cot x| + c = \log \tan\left(\frac{x}{2}\right) + c$

(xi) $\int \sec x \tan x\, dx = \sec x + c$

(xii) $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + c$

(xiii) $\int \sec^2 x\, dx = \tan x + c$

(xiv) $\int \text{cosec}^2 x\, dx = -\cot x + c$

(xv) $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

(xvi) $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c \quad (x > a)$

(xvii) $\int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c \quad (x < a)$

(xviii) $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right)+c = -\cos^{-1}\left(\frac{x}{a}\right)+c$

(xix) $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log|x+\sqrt{x^2+a^2}|+c = \sinh^{-1}\left(\frac{x}{a}\right)+c$

(xx) $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log|x+\sqrt{x^2-a^2}|+c = \cosh^{-1}\left(\frac{x}{a}\right)+c$

(xxi) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+c$

(xxii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}+c$

(xxiii) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}+c$

(xxiv) $\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a}+c$

(xxv) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left\{bx-\tan^{-1}\left(\frac{b}{a}\right)\right\}+c$

(xxvi) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)+c = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left\{bx-\tan^{-1}\left(\frac{b}{a}\right)\right\}+c$

(xxvii) $\int e^{ax+b}(af(x)+f'(x))dx = e^{ax+b}f(x)+c$

(xxviii) $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b)+c$

METHOD OF INTEGRATION

Integration by Substitution

(a) When integrand is the product of two factors such that one is the derivative of the other i.e,

$$I = \int f(x)\,f'(x)dx$$

In this case we put $f(x) = t$ to convert it into a standard integral.

(b) When integrand is a function of function i.e. $\int f[\phi(x)]\phi'(x)\ dx$

Here we put f(x) = t so that f ' (x) dx = dt and in that case the integrand is reduced to $\int f(t)\,dt$.

(c) Integral of a function of the form (ax+b) dx

Here put ax + b = t and convert it into standard integral. Obviously if $\int f(x)\,dx = \phi(x)$, then

$$\int f(ax+b)\ dx = \frac{1}{a}\phi\ (ax+b)$$

(d) Some standard forms of integrals

The following three forms are very useful to write integral directly.

(i) $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$ (where n ¹ –1)

(ii) $\int \frac{f'(x)}{f(x)}dx = \log[f(x)] + c$

(iii) $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$

(e) Integral of the form $\int \frac{dx}{a\sin x + b\cos x}$

Putting a = r cos θ and b = r sin θ, we get

$$I = \int \frac{dx}{r\sin(x+\theta)} = \frac{1}{r}\int \text{cosec}\,(x+\theta)\,dx$$

$$= \frac{1}{r}\log\tan(x/2+\theta/2) + c \quad = \int \frac{1}{\sqrt{a^2+b^2}}\log\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{b}{a}\right) + c$$

(f) Standard Substitution

Following standard substitution will be useful-

Integrand form	Substitution
(i) $\sqrt{a^2-x^2}$ or $\frac{1}{\sqrt{a^2-x^2}}$	$x = a\sin\theta$
(ii) $\sqrt{x^2+a^2}$ or $\frac{1}{\sqrt{x^2+a^2}}$	$x = a\tan\theta$ or $x = a\sinh\theta$
(iii) $\sqrt{x^2-a^2}$ or $\frac{1}{\sqrt{x^2-a^2}}$	$x = a\sec\theta$ or $x = a\cosh\theta$
(iv) $\sqrt{\frac{x}{a+x}}$ or $\sqrt{\frac{a+x}{x}}$ or $\sqrt{x(a+x)}$ or $\frac{1}{\sqrt{x(a+x)}}$	$x = a\tan^2\theta$
(v) $\sqrt{\frac{x}{a-x}}$ or $\sqrt{\frac{a-x}{x}}$ or $\sqrt{x(a-x)}$ or $\frac{1}{\sqrt{x(a-x)}}$	$x = a\sin^2\theta$

(vi) $\sqrt{\frac{x}{x-a}}$ or $\sqrt{\frac{x-a}{x}}$ or $\sqrt{x(x-a)}$ or $\frac{1}{\sqrt{x(x-a)}}$ $\qquad x = a\sec^2\theta$

(vii) $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$ $\qquad x = a\cos 2\theta$

(viii) $\sqrt{\frac{x-\alpha}{\beta-x}}$ or $\sqrt{(x-\alpha)(\beta-x)}$ (b > a) $\qquad x = a\cos^2\theta + b\sin^2 q$

(a) Integration by Parts :

If u and v are the differentiable functions of x, then $\int u.v\ dx = u\int v dx - \int\left[\left(\frac{d}{dx}(u)\right)\left(\int v dx\right)\right] dx.$

i.e. Integral of the product of two functions = first function x integral of second function – $\int$ [derivative of first) x (Integral of second)]

(i) How to choose Ist and IInd function : If two functions are of different types take that function as Ist which comes first in the word ILATE, where I stands for inverse circular function, L stands for logrithmic function, A stands for algebric functions, T stands for trigonometric and E for exponential functions.

(ii) For the integration of logarthmic or inverse trigonometric functions alone, take unity (1) as the second function

(b) If the integral is of the form $\int e^x[f(x)+f'(x)]\,dx$ then by breaking this integral into two integrals, integrate one integral by parts and keep other integral as it is, By doing so, we get - $\int e^x[f(x)+f'(x)]dx = e^x f(x) + c$

(c) If the integral is of the form $\int [x\,f'(x)+f(x)]\,dx$ then by breaking this integral into two integrals integrate one integral by parts and keep other integral as it is, by doing so, we get $\int [x\,f'(x)+f(x)]\,dx = x f(x) + c$

Integration of the Trigonometrical Functions

(i) $\int \frac{dx}{a+b\sin^2 x}$, $\qquad$ (ii) $\int \frac{dx}{a\cos^2 x + b}$

(iii) $\int \frac{dx}{a\cos^2 x + b\sin^2 x}$, $\qquad$ (iv) $\int \frac{dx}{(a\cos x + b\sin x)^2}$.

(For their integration we multiply and divide by $\sec^2 x$ and then put tan x = t.)

Some integrals of different expressions of e^x

(i) $\int \frac{ae^x}{b+ce^x}dx$ $\qquad$ [put $e^x = t$]

(ii) $\int \frac{1}{1+e^x}dx$ $\qquad$ [multiplying and divide by e^{-x} and put $e^{-x} = t$]

(iii) $\int \frac{1}{1-e^x}dx$ $\qquad$ [multiplying and divide by e^{-x} and put $e^{-x} = t$]

(iv) $\int \frac{1}{e^x - e^{-x}}dx$ $\qquad$ [multiply and divided by e^x]

(v) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$ $\left[\frac{f'(x)}{f(x)} \text{form}\right]$

(vi) $\int \frac{e^x + 1}{e^x - 1} dx$ [multiply and divide by $e^{-x/2}$]

(vii) $\int \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 dx$ [integrand = $\tanh^2 x$]

(viii) $\int \left(\frac{e^{2x} + 1}{e^{2x} - 1}\right)^2 dx$ [integrand = $\coth^2 x$]

(ix) $\int \frac{1}{(e^x + e^{-x})^2} dx$ [integrand = $\frac{1}{4}$ $\text{sech}^2 x$]

(x) $\int \frac{1}{(e^x - e^{-x})^2} dx$ [integrand = $\frac{1}{4}$ $\text{cosech}^2 x$]

(xi) $\int \frac{1}{(1+e^x)(1-e^{-x})} dx$ [multiply and divide by e^x and put $e^x = t$]

(xii) $\int \frac{1}{\sqrt{1-e^x}} dx$ [multiply and divide by $e^{-x/2}$]

(xiii) $\int \frac{1}{\sqrt{1+e^x}} dx$ [multiply and divide by $e^{-x/2}$]

(xiv) $\int \frac{1}{\sqrt{e^x - 1}} dx$ [multiply and divide by $e^{-x/2}$]

(xv) $\int \frac{1}{\sqrt{2e^x - 1}} dx$ [multiply and divide by $\sqrt{2}e^{-x/2}$]

(xvi) $\int \sqrt{1-e^x} dx$ [integrand = $(1 - e^x) / \sqrt{1-e^x}$]

(xvii) $\int \sqrt{1-e^x} dx$ [integrand = $(1 + e^x) / \sqrt{1+e^x}$]

(xviii) $\int \sqrt{e^x - 1}\ dx$ [integrand = $(e^x - 1) / \sqrt{e^x - 1}$]

(xix) $\int \sqrt{\frac{e^x + a}{e^x - a}}\ dx$ [integrand = $(e^x + a) / \sqrt{e^{2x} - a^2}$]

8. $\int \frac{x^2 + 1}{x^4 + kx^2 + 1} dx$ (Divide N.r and Dr by x^2 then put $x \pm 1/x = t$)

9. $\int \frac{x^2 - 1}{x^4 + kx^2 + 1} dx$ (Divide N.r and Dr by x^2 then put $x \pm 1/x = t$)

$N^r = A\,(D^r) + B\,\frac{d}{dx}(D^r) + C$

10. $\int \frac{x^2}{x^4 + kx^2 + 1}dx$ $\qquad x^2 = \frac{1}{2}\{(x^2+1) + (x^2-1)\}$

11. $\int \frac{1}{x^4 + kx^2 + 1}dx$ $\qquad 1 = \frac{1}{2}\{(x^2+1) - (x^2-1)\}$

12. $\int \frac{1}{x^4 + a^4}dx$ $\qquad 1 = \frac{1}{2a^2}\{(x^2+a^2) - (x^2-a^2)\}$

13. $\int \frac{1}{(ax+b)\sqrt{cx+d}}dx$; $\qquad$ Put $(x + d) = t^2$

14. $\int \frac{1}{(px+q)\sqrt{ax^2+bx+c}}dx$; $\qquad$ Put $(px + q) = \frac{1}{t}$

15. $\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}}dx$; $\qquad$ Put $(px + q) = t^2$

16. $\int \frac{1}{(Ax^2+B)\sqrt{cx^2+D}}dx$; $\qquad$ Put $(x = 1/t)$

17. $\int \frac{1}{(a\sin^2 x + b\sin x\cos x + c.\cos^2 x)}dx$

18. $\int \frac{1}{(a+b\sin x)}dx$; put $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$

19. $\int \frac{1}{(a+b\cos x)}dx$ put $\cos x = \left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)$ $\qquad$ & put $\tan x/2 = t$

20. $\int \frac{1}{(a\sin x + b\cos x + c)}dx$

21. $\int \frac{P\sin x + q\cos x + r}{a\sin x + b\cos x + c}dx$